9.3 Exponential / logarithm

differentiation rules:

$$\frac{d}{dx}(\text{const}) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$ power rule ($n \in \mathbb{Q}$)

polynomials

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}(kf) = k\frac{df}{dx}$$

product rule

chain rule

$$\frac{d}{dx}(fg) = f'g + fg'$$

$$\frac{d}{dx}(f/g) = \frac{d}{dx}\left(f(g)^{-1}\right) = f'(g)^{-1} + f\cdot(-1)(g)^{-2}\cdot g'$$

$$= \frac{f'}{g} - \frac{fg'}{g^2}$$

$$= \frac{f'g - fg'}{g^2}$$ quotient rule

$$(f\circ g)'(x) = f'(g(x))\,g'(x)$$ chain rule

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$x^3 \longrightarrow 3x^2$$

$$x^2 \longrightarrow 2x^1$$

$$x^1 \longrightarrow 1 \cdot x^0$$

$$x^0 \longrightarrow 0$$

where is x^{-1}?

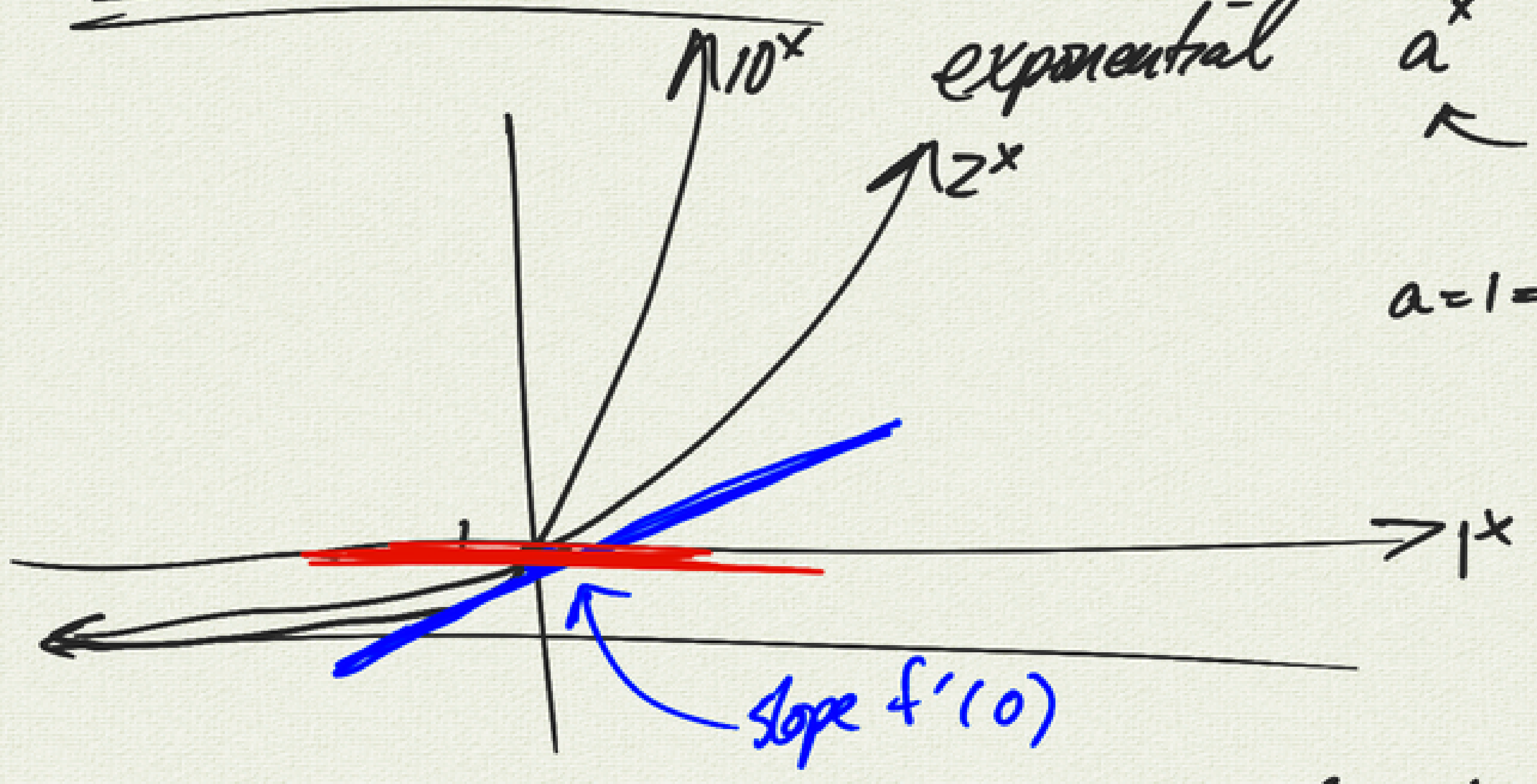
$$x^{-1} \longrightarrow -1x^{-2} = -\frac{1}{x^2}$$

$$x^{-2} \longrightarrow -2x^{-3}$$

Story of e

exponential a^x $\leftarrow a=2,10$

$a=1 \Rightarrow 1^x$



observation: increase a $\Rightarrow$ increase slope (at 0)

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{a^{0+h}-a^0}{h}$$
$$=\lim_{h\to 0}\frac{a^h-1}{h} = ?$$

call this $l(a)$: $l(1)=0$

l increasing

what about base $(ab)^x$ $\Leftarrow$ relationship to a^x, b^x

$$l(ab) \overset{?}{\Leftarrow} l(a),\ l(b)$$

$l(ab)=$ slope at 0 of $(ab)^x$
$$=\lim_{h\to 0}\frac{(ab)^{0+h}-(ab)^0}{h}$$
$$=\lim_{h\to 0}\frac{\boxed{(ab)^h-1}}{h}$$
$$=\lim_{h\to 0}\underbrace{\frac{(a^h-1)(b^h-1)}{h}}_{\to 0}+\underbrace{\frac{(a^h-1)}{h}}_{l(a)}+\underbrace{\frac{(b^h-1)}{h}}_{l(b)}$$

$$(a^h-1)(b^h-1)$$
$$=a^hb^h-a^h-b^h+1$$
$$\Rightarrow a^hb^h-1=(a^h-1)(b^h-1)+(a^h-1)+(b^h-1)$$

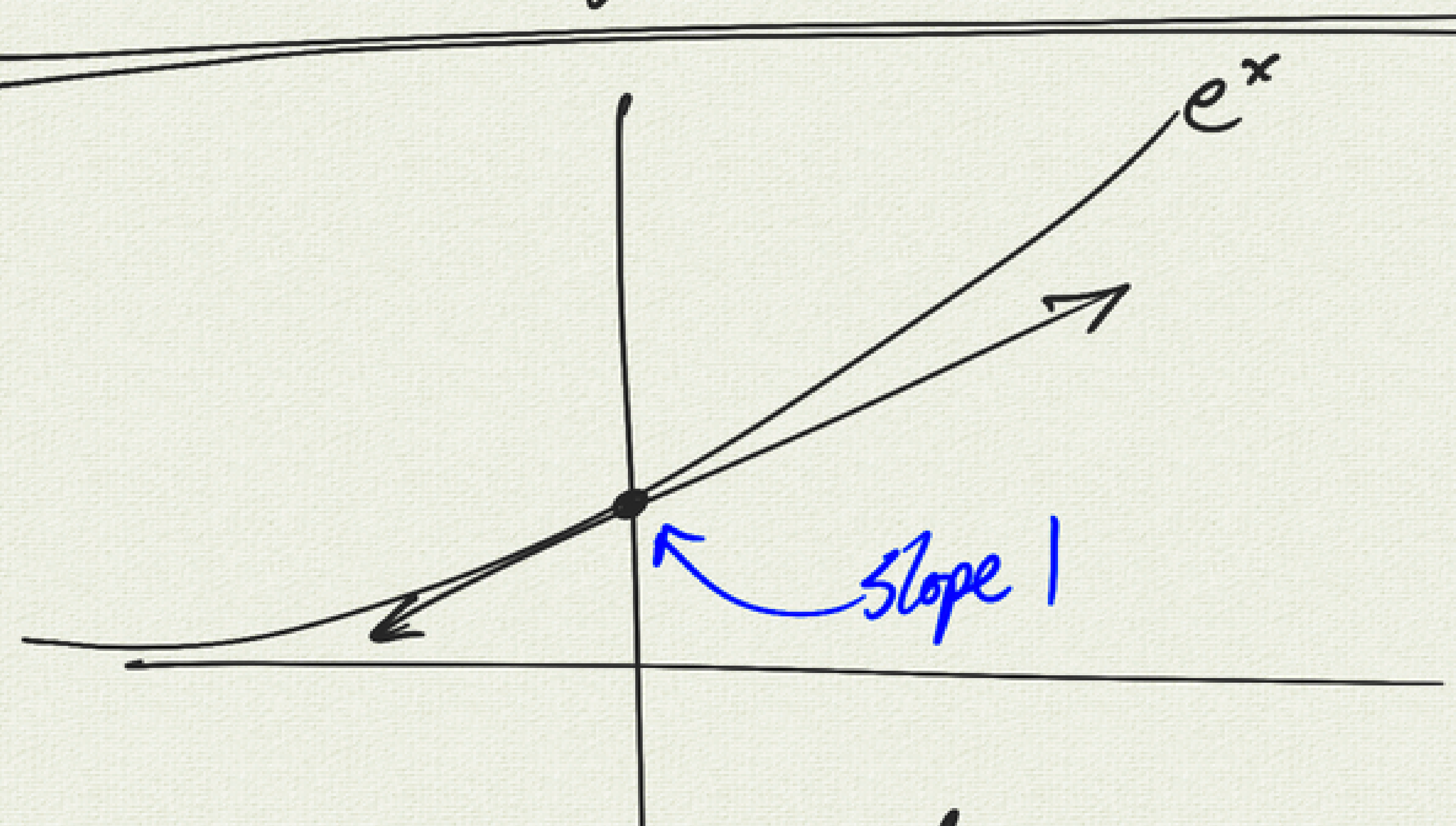
$$\boxed{l(ab)=l(a)+l(b)}$$

logarithm

$\log(ab)=\log(a)+\log(b)$

$\Rightarrow$ what is the base?

$$l(a)=\log_e(a) \Rightarrow l(e)=\log_e(e)=1$$



$$1=\frac{d}{dx}(e^x)(0)=\lim_{h\to 0}\frac{e^{0+h}-e^0}{h}$$
$$\boxed{1=\lim_{h\to 0}\frac{e^h-1}{h}}$$ special limit

$$l(a)=\boxed{\ln(a)=\lim_{h\to 0}\frac{a^h-1}{h}}$$

$$\frac{d}{dx}(e^x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$
$$=\lim_{h\to 0}\frac{e^xe^h-e^x}{h}$$
$$=\lim_{h\to 0}e^x\left(\frac{e^h-1}{h}\right)$$
$$=e^x\underbrace{\lim_{h\to 0}\frac{e^h-1}{h}}_{\to 1}$$

replace e with a

$$\boxed{\frac{d}{dx}(e^x)=e^x}$$

$$\frac{d}{dx}(a^x)=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$
$$=a^x\underbrace{\lim_{h\to 0}\frac{a^h-1}{h}}_{\ln(a)}$$

$$\boxed{\frac{d}{dx}(a^x)=a^x\ln a}$$

example

product rule: $0 \cdot e^x + 5 \cdot e^x$

$$f(x) = 5e^x + 10 \cdot 2^x$$

$$f'(x) = 5e^x + 10(2^x \ln 2)$$

$$g(x) = e^{\sin x} = (h \circ g)(x)$$

$h(x) = e^x$
$g(x) = \sin x$
chain rule

$$\Rightarrow g'(x) = e^{\sin x} \cdot \cos x$$

$$h'(g(x)) \quad g'(x)$$

$$h(x) = \cos^3(e^{x^2})$$

inner
square $\to$ exp
$\to$ cos
$\to$ cube
outer

$$= [\cos(e^{x^2})]^3$$

$$= 3[\cos(e^{x^2})]^2 \cdot (-\sin(e^{x^2})) \cdot e^{x^2} \cdot 2x$$

$y = \ln x \Rightarrow \frac{dy}{dx} = ?$

$\Downarrow$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$$\boxed{\frac{d(\ln x)}{dx} = \frac{1}{x} = x^{-1}}$$

There it is!

$y = \log_a x \Rightarrow a^y = x$

$a^y \ln a \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$

$$\boxed{\frac{d(\log_a x)}{dx} = \frac{1}{x \ln a}}$$

$y = x^n \leftarrow n \in \mathbb{R}$

$\ln y = \ln(x^n)$

$\ln y = n \ln x$

$\frac{1}{y} \frac{dy}{dx} = n \frac{1}{x}$

$\frac{dy}{dx} = n \frac{y}{x} = \frac{n x^n}{x} = n x^{n-1}$ ⇐ power rule works for any $n \in \mathbb{R}$

$\ln(ab) = \ln(a) + \ln(b)$

$\ln(a^n) = n \ln a$

Summary

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \ln a \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$\approx 2.71828$$

HW

(303) $3x^3 + 9xy^2 = 5x^3$

$$\Rightarrow \underline{\underline{9x^2}} + 9\Big(1\cdot y^2 + x\cdot \underbrace{2y\frac{dy}{dx}}_{\frac{d}{dx}(y^2)}\Big) = 15x^2$$