KEY

Unit 3 Group Work
PCHA 2021-22 / Dr. Kessner

Name / Pledge:

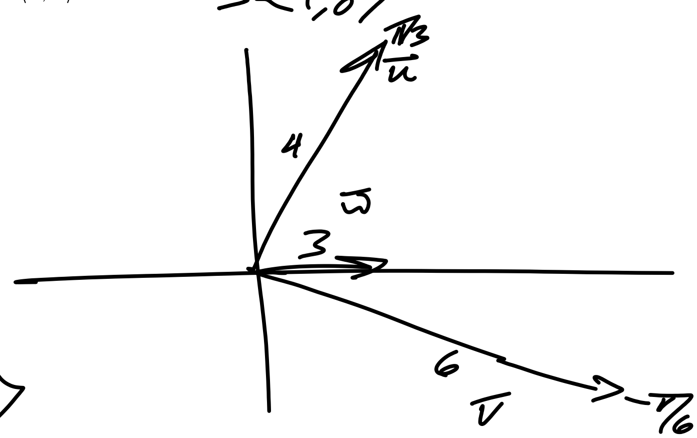
Partner(s):

You can use your notes and/or textbook. No calculator. Have fun!

1. Suppose you have the following vectors:

$$\vec{u} = \left\langle 2, 2\sqrt{3} \right\rangle = 4\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$
$$\vec{v} = \left\langle 3\sqrt{3}, -3 \right\rangle = 6\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$
$$\vec{w} = \langle 3, 0 \rangle = 3\langle 1, 0 \rangle$$

Calculate the following:



a) $|\vec{u}|$ 4

b) $|\vec{v}|$ 6

c) Unit vector in the direction of $\vec{v}$. $\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

d) Angle between $\vec{u}$ and $\vec{v}$. $\pi/2$

e) Angle between $\vec{u}$ and $\vec{w}$. $\pi/3$

2. a) Parametrize the line segment from $(1,2)$ to $(3,6)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

b) Parametrize the line segment from $(3,6)$ to $(1,2)$ (same points, opposite direction).

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} + t \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

c) Parametrize the circle with center $(3,4)$ and radius 5.

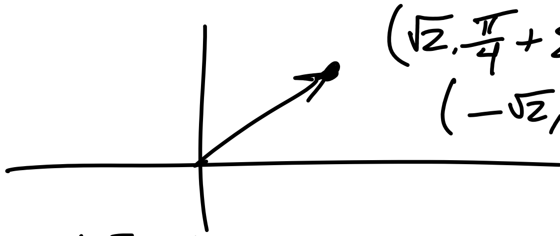
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5\cos t \\ 5\sin t \end{pmatrix}$$

d) Parametrize the same circle, but make the period $= 6$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5\cos \frac{2\pi}{6} t \\ 5\sin \frac{2\pi}{6} t \end{pmatrix}$$

3. Find all polar coordinates of the following (rectangular) points:

a) (1, 1)



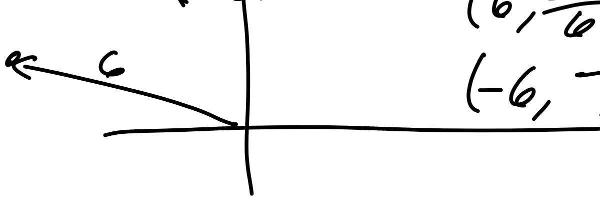
$(\sqrt{2}, \frac{\pi}{4} + 2\pi k)$

$(-\sqrt{2}, \frac{5\pi}{4} + 2\pi k)$

b) $(-3\sqrt{3}, 3)$ $= 6\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$(6, \frac{5\pi}{6} + 2\pi k)$

$(-6, -\frac{\pi}{6} + 2\pi k)$



Convert the following equations from rectangular to polar coordinates:

c) $3x + 4y = 5$

$3r\cos\theta + 4r\sin\theta = 5$

$r = \dfrac{5}{3\cos\theta + 4\sin\theta}$

d) $x^2 + y^2 = 25$

$r = 5$

Convert from polar to rectangular:

e) $r = -5\sin\theta$

$r^2 = -5r\sin\theta$

$x^2 + y^2 + 5y = 0$

$x^2 + (y + \frac{5}{2})^2 = (\frac{5}{2})^2$

f) $r = 5\csc\theta$

$r\sin\theta = 5$

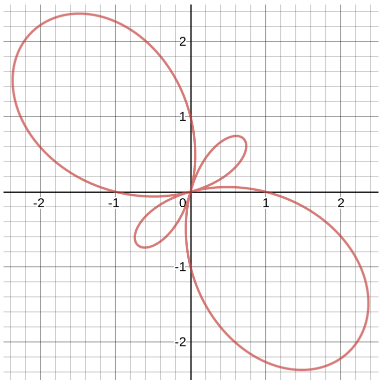
$y = 5$

4. Analyze the graph of the polar function $r = 1 - 2\sin 2\theta$:

1) Find the max $|r|$ values and θ values where they occur.

2) State and prove any symmetry relations.

3) **Challenge:** What is going on at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$?



① $\max|r| = \max|1 - 2\sin 2\theta|$

$= 3$ when $\sin 2\theta = -1$

$2\theta = \frac{3\pi}{2} + 2\pi k$

$\theta = \frac{3\pi}{4} + \pi k$

② No x-axis or y-axis symmetry (by inspection of graph)

origin symmetry:

$(-r, \theta)$:

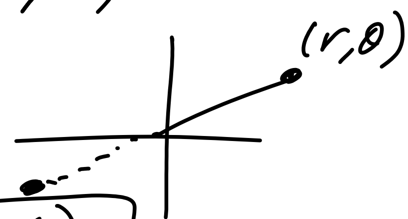
$-r \stackrel{?}{=} 1 - 2\sin 2\theta$ ✗

$(r, \theta + \pi)$:

$r \stackrel{?}{=} 1 - 2\sin 2(\theta + \pi)$

$= 1 - 2\sin(2\theta + 2\pi)$

$= 1 - 2\sin 2\theta$ ✓



origin test: $\boxed{\begin{array}{c}(-r, \theta)\\ (r, \theta + \pi)\end{array}}$

③ $r = -1$ at $\pi/4$, $5\pi/4$

this is a min for r,
and a local max for $|r|$
occuring at $\theta = \pi/4, 5\pi/4$

5. For each of the following 2x2 matrices, determine whether it is invertible, and if so, find the inverse matrix and the determinant of the inverse.

$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad |A| = 9 \quad A^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \frac{1}{3}I \quad \det A^{-1} = \frac{1}{9}$

$B = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad |B| = -4 \quad B^{-1} = -\frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \det B^{-1} = -\frac{1}{4}$

$= \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

$C = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ $\quad |C| = -4 \quad C^{-1} = -\frac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \quad \det C^{-1} = -\frac{1}{4}$

$= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad |D| = 0$

not invertible

Let $E = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}$. Find E^{-1}. Verify that $EE^{-1} = I$.

$EE^{-1} = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$|E| = -1 \quad E^{-1} = -\begin{pmatrix} 4 & -5 \\ -5 & 6 \end{pmatrix}$

$= \begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix}$

Use the inverse matrix you found to solve the following linear systems:

$6x + 5y = 1$
$5x + 4y = 0$

$\begin{pmatrix} x \\ y \end{pmatrix} = E^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$6x + 5y = 0$
$5x + 4y = 1$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$

$6x + 5y = 1$
$5x + 4y = 2$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 5 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}$

6. Consider the following system of linear equations:

$$\begin{aligned} x + 3z &= 4 \\ -x - 2z &= -3 \\ y - 2z &= -1 \end{aligned}$$

a. Write the linear system as a matrix equation.

$$\underbrace{\begin{pmatrix} 1 & 0 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$$

b. Calculate the determinant of the matrix to verify that the matrix is invertible.

$$\det A = \begin{vmatrix} 0 & -2 \\ 1 & -2 \end{vmatrix} - 0 \begin{vmatrix} & \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 2 - 3$$

$$= -1 \checkmark$$

invertible

c. Find the inverse matrix and use it to solve the system.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$R_1 + R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

R_{23}

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$-3R_3 + R_1$
$2R_3 + R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$A^{-1}\begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & -3 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -8+9 \\ 8-6-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$