Unit 4 Group Work 2
PCHA 2021-22 / Dr. Kessner

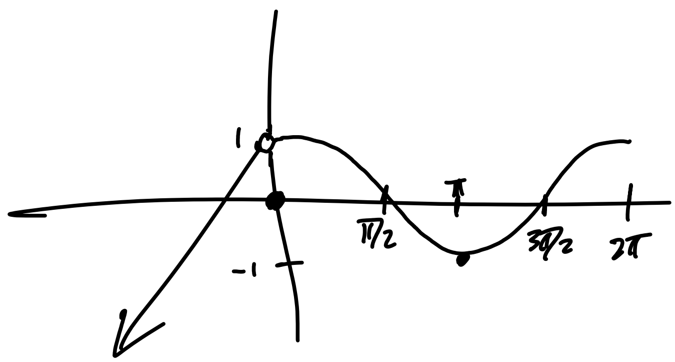
KEY

No calculator! Have fun!

1. Let

$$f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \cos x & \text{if } x > 0 \end{cases}$$

a) Sketch the graph of $f(x)$.



b) On what intervals is f increasing and/or decreasing? Is f bounded? Does it have any local or global maxima or minima?

increasing on $(-\infty, 0)$
increasing on $[\pi + 2\pi k, 2\pi + 2\pi k]$
decreasing on $[0 + 2\pi k, \pi + 2\pi k]$
$k \in \mathbb{Z}^+$

global + local max at $2\pi + 2\pi k$
local min at $\pi + 2\pi k$
$k \in \mathbb{Z}^+$
$k = 0, 1, 2, \ldots$
bounded above

c) Does f have any discontinuities? Where, and what type?

removable discontinuity at $x = 0$

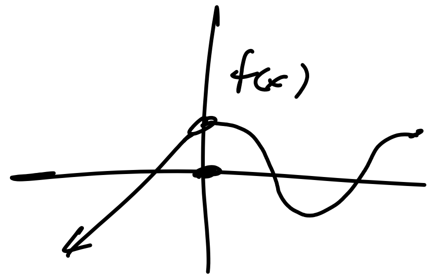
d) Describe the end behavior of f using limits.

$\lim_{x \to -\infty} f(x) = -\infty$

$\lim_{x \to \infty} f(x)$ does not exist

2. Consider the same function from the previous problem.

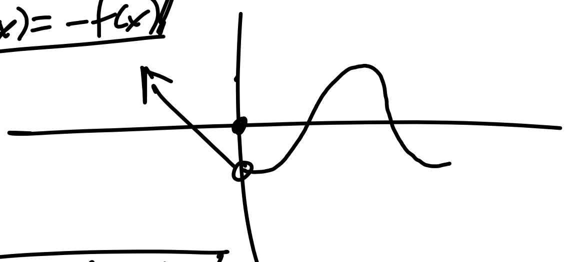
$$f(x) = \begin{cases} x+1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ \cos x & \text{if } x>0 \end{cases}$$



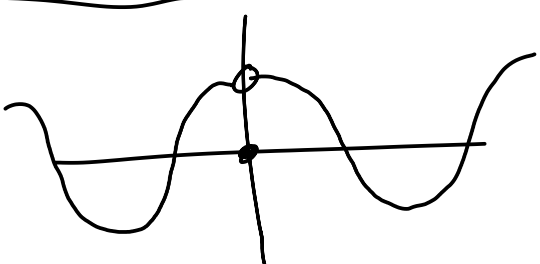
Sketch the graphs of the following transformed functions:

- $p(x) = -f(x)$
- $q(x) = f(|x|)$
- $r(x) = -f(|x|)$
- $s(x) = f(-|x|)$
- $t(x) = |f(-|x|)|$

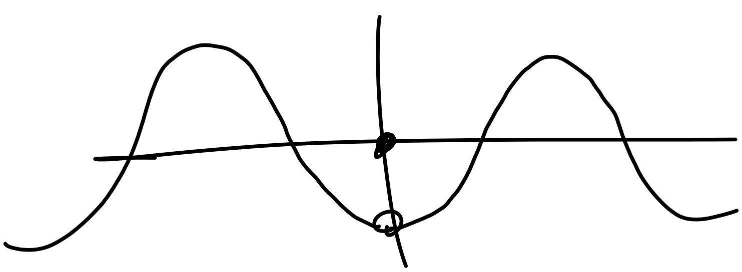
$p(x) = -f(x)$



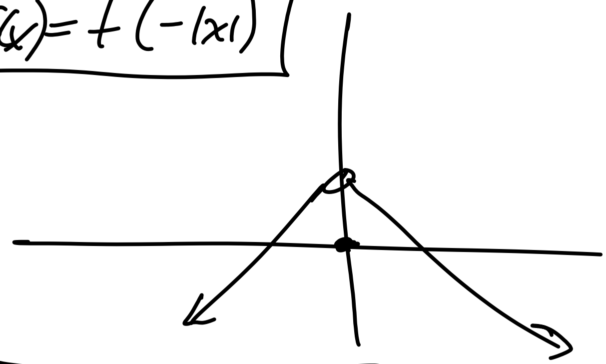
$q(x) = f(|x|)$



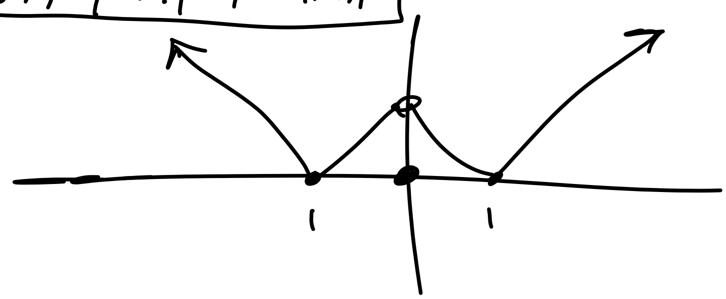
$r(x) = -q(x) = -f(|x|)$



$s(x) = f(-|x|)$



$t(x) = |s(x)| = |f(-|x|)|$



3. Factor the following polynomial completely, both over $\mathbb{R}$ (as a product of real linear and irreducible quadratic factors) and over $\mathbb{C}$ (as a product of complex linear factors). Sketch the graph of the function.

$$p(x) = x^4 + 6x^3 + 13x^2 + 12x + 4$$

potential rational roots: $\pm 1, 2, 4$

$p(1) \neq 0$

$p(-1) = +1 - 6 + 13 - 12 + 4 = 0$ ✓

$$\begin{array}{r|rrrrr} -1 & 1 & 6 & 13 & 12 & 4 \\ & & -1 & -5 & -8 & -4 \\ \hline -1 & 1 & 5 & 8 & 4 & \boxed{0} \\ & & -1 & -4 & -4 & \\ \hline & 1 & 4 & 4 & \boxed{0} & \end{array}$$

$\Rightarrow p(x) = (x+1)\underbrace{(x^3 + 5x^2 + 8x + 4)}_{q(x)}$

$q(-1) = -1 + 5 - 8 + 4 = 0$ ✓

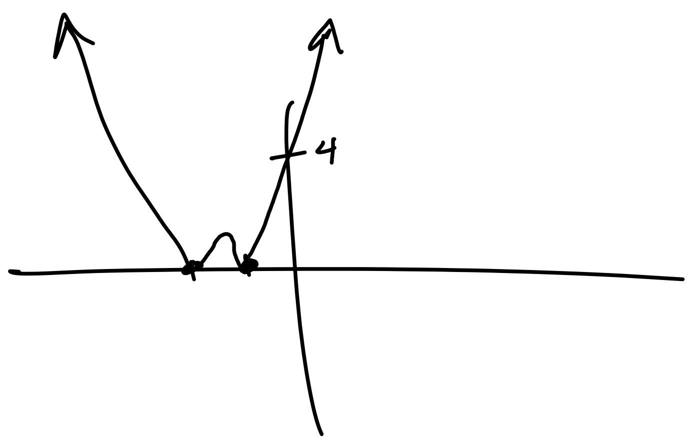
(-1 is a double root)

$x^2 + 4x + 4$

$(x+2)^2$

$\Rightarrow p(x) = (x+1)^2(x+2)^2$

(over $\mathbb{R}$ or $\mathbb{C}$)



4. Factor the following polynomial completely, both over $\mathbb{R}$ (as a product of real linear and irreducible quadratic factors) and over $\mathbb{C}$ (as a product of complex linear factors). Sketch the graph of the function.

$$q(x) = x^5 + 2x^4 - 16x - 32$$

potential rational roots: $\pm 1, 2, 4, 8, 16, 32$

$q(1) = 1+2-16-32 \neq 0$

$q(-1) = -1+2+16-32 \neq 0$

$q(2) = 32+32-32-32 = 0$ ✓

$$\begin{array}{r|rrrrrr}
2 & 1 & 2 & 0 & 0 & -16 & -32 \\
 & & 2 & 8 & 16 & 32 & 32 \\ \hline
-2 & 1 & 4 & 8 & 16 & 16 & \boxed{0} \\
 & & -2 & -4 & -8 & -16 & \\ \hline
-2 & 1 & 2 & 4 & 8 & \boxed{0} & \\
 & & -2 & 0 & -8 & & \\ \hline
 & 1 & 0 & 4 & \boxed{0} & &
\end{array}$$

$q(x) = (x-2)\underbrace{(x^4+4x^3+8x^2+16x+16)}_{r(x)}$

$r(2) \neq 0$

$r(-2) = 16-32+32-32+16 = 0$ ✓

$q(x) = (x-2)(x+2)\underbrace{(x^3+2x^2+4x+8)}_{s(x)}$

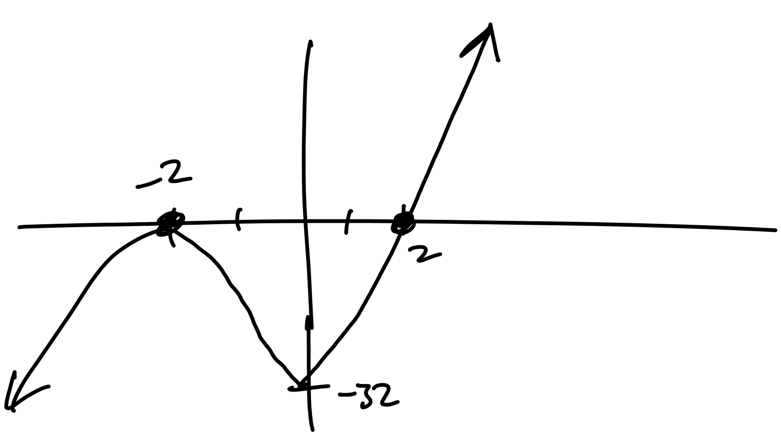
$s(-2) = -8+8-8+8 = 0$ ✓

x^2+4

$(x+2i)(x-2i)$

$\Rightarrow$ $q(x) = (x-2)(x+2)^2(x^2+4)$ over $\mathbb{R}$

$= (x-2)(x+2)^2(x+2i)(x-2i)$ over $\mathbb{C}$



5. Sketch the graph of the following rational function.

$$r(x) = \frac{x^3 + x^2 - x - 1}{x}$$

Write limits to describe its end behavior, and its behavior near asymptotes. Challenge: Describe its asymptotic end behavior.

factor $p(x) = x^3 + x^2 - x - 1$

$p(1) = 0 \Rightarrow$

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -1 & -1 \\ & & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$x^2 + 2x + 1 = (x+1)^2$

$p(x) = (x-1)(x+1)^2$

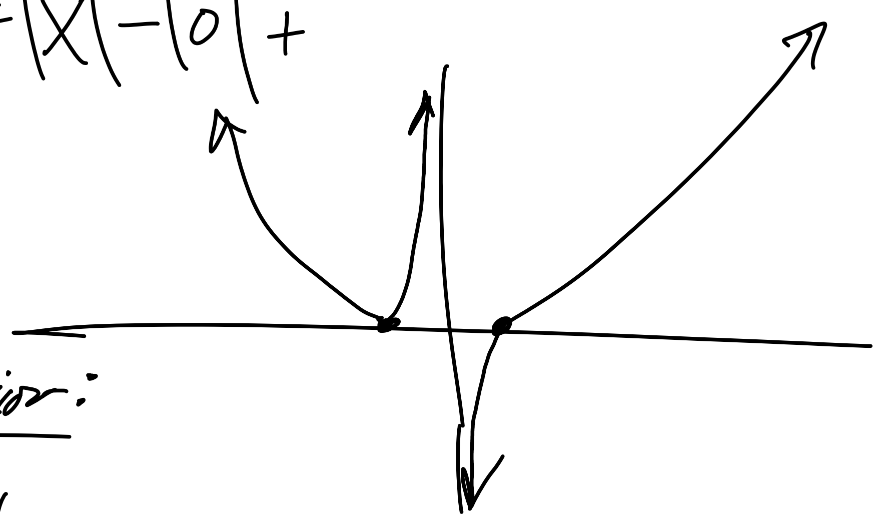
$r(x) = \dfrac{(x-1)(x+1)^2}{x}$ ← zeros at ± 1

← asymptote at $x = 0$

		-1		0		1	
$(x+1)^2$	$+$	0	$+$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$r(x)$	$+$	0	$+$	X	$-$	0	$+$

end behavior

$\lim_{x \to \pm\infty} r(x) = \infty$



asymptotic end behavior:

$r(x) = \dfrac{x^3 + x^2 - x - 1}{x}$

$$\begin{array}{r|rrrr} 0 & 1 & 1 & -1 & -1 \\ & & 0 & 0 & 0 \\ \hline & 1 & 1 & -1 & \boxed{-1} \end{array}$$

$r(x) = x^2 + x - 1 - \frac{1}{x}$

$r(x) \approx x^2 + x - 1$

as $x \to \infty$

asymptotically parabolic