**Discrete Math Practice**
**PCHA 2021-22 / Dr. Kessner**

KEY

**No calculator unless its an emergency. Have fun!**

**1.** Evaluate:

a. $\binom{8}{0} = 1$

b. $\binom{8}{1} = 8$

c. $\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$

d. $\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3!} = 56$

e. $\binom{8}{6} = \binom{8}{2} = 28$

f. You have 8 dogs and 5 cats. How many ways can you pick a committee of 3 dogs and 2 cats?

$\binom{8}{3}\binom{5}{2} = 56 \cdot 10 = 560$ $\qquad \binom{5}{2} = \frac{5 \cdot 4}{2} = 10$

g. You still have 8 dogs and 5 cats. How many ways can you rank your top 3 cats and top 3 dogs?

$\underbrace{8 \cdot 7 \cdot 6}_{{}_8P_3} \cdot \underbrace{5 \cdot 4 \cdot 3}_{{}_5P_3}$

**2.** Phone numbers have the format `(xxx)yyy-zzzz`. For both the area code `xxx` and exchange `yyy`, the first digit is not allowed to be 0 or 1.

a. How many possible phone numbers are there?

(xxx) yyy - zzzz

$8 \cdot 10 \cdot 10 \quad 8 \cdot 10 \cdot 10 \cdot \quad 10 \cdot 10 \cdot 10 \cdot 10 = 64 \cdot 10^8$

$= 6.4 \times 10^9$

6.4 billion

b. Some phone numbers are not available for standard telephones. In particular, the numbers 555-0100 through 555-0199 (in any area code) are reserved for fictional use. In addition, the N11 numbers (e.g. 911, 411) are also reserved, so these are not usable for the exchange `yyy`. Considering these two restrictions, how many phone numbers are available for telephones?

(xxx) yyy - zzzz

$(8 \cdot 10 \cdot 10)\, 1 \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 10 \cdot 10 = 8 \cdot 10^4$ "555" numbers

555 0122

$(8 \cdot 10 \cdot 10)\, 8 \cdot 1 \cdot 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 16 \cdot 10^6 = 1.6 \times 10^7$ 411/911 numbers

y 11 zzzz

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leftover: $6.4 \times 10^9 - 64 \times 10^7 - 8 \times 10^4$

**3.** You have 11 possible toppings you could put on your pizza.

a. How many different pizzas could you make?

$2^{11} = 2048$

b. Your favorite number is 9. How many different pizzas can you make using exactly 9 of the 11 toppings?

$\binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55$

c. You pick a random pizza from the set of all possible different pizzas. What is the probability that it has exactly 9 toppings?

$\frac{\binom{11}{9}}{2^{11}} = \frac{55}{2048}$

d. You pick a random pizza from the set of all possible different pizzas. What is the probability that it has exactly 10 toppings?

$\frac{\binom{11}{10}}{2^{11}} = \frac{11}{2048}$

**5.** a. Expand and write in standard form: $(-3x+y)^4$.

$$(-3x+y)^4 = (-3x)^4 + 4(-3x)^3(y)^1 + 6(-3x)^2y^2 + 4(-3x)^1y^3 + y^4$$
$$= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$$

b. Find the coefficient of the $x^2y^6$ term in the expansion of $(-2y+x)^8$.

$\binom{8}{2} = \frac{8\cdot 7}{2} = 28$

$x^2y^6$ term:

$$\binom{8}{2}(x^2)(-2y)^6$$
$$= 28 \cdot 64\, x^2y^6$$

**5.** a. How many ways can you choose a random 5-letter word? Assume that the alphabet has the usual 26 letters, and a word is any sequence of letters, whether you can pronounce it or not.

$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^5$

b. Suppose you pick a random 5 letter word. What is the probability of picking the word `PROOF`?

$\frac{1}{26^5}$

c. Assume that the letter `y` is a vowel, so there are 6 vowels and 20 consonants in the alphabet. What is the probability of picking a word having the form `XXOOX`, where `X` is a consonant and `O` is a vowel?

$\frac{20}{X} \cdot \frac{20}{X} \cdot \frac{6}{O} \cdot \frac{6}{O} \cdot \frac{20}{X} = \frac{20^3 6^2}{26^5} \approx .024$

d. What is the probability that a random 5-letter word has exactly 2 vowels? *Hint:* In part (c) you calculate the probability of picking a word with vowels in the 3rd and 4th positions. But a random word may have exactly 2 vowels in some other positions. How many ways can you pick 2 positions to be vowels from the 5 positions?

$\binom{5}{2} \cdot \frac{20^3 6^2}{26^5} \approx .24$