Unit 8 Group Work
PCHA 2021-22 / Dr. Kessner

KEY

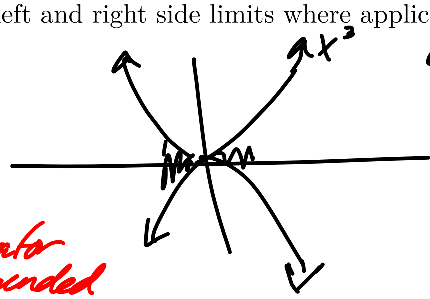
No calculator! Have fun!

1. Evaluate the following limits, evaluating left and right side limits where applicable.

a. $\lim\limits_{x\to 0} x^3 \cos x = 0$

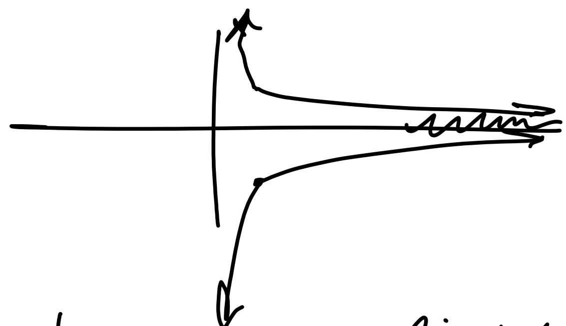
$x^3 \to 0$, $\cos x \to 1$

or squeeze Theorem

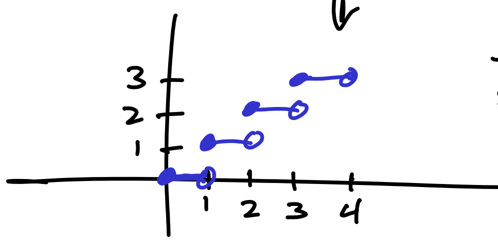


numerator is bounded

b. $\lim\limits_{x\to\infty} \dfrac{\sin 2x + \cos 3x + 5}{x} = 0$



c. $\lim\limits_{x\to 3} \text{int}(x)$ (the greatest integer function)



$\lim\limits_{x\to 3^-} \text{int}(x) = 2$

$\lim\limits_{x\to 3^+} \text{int}(x) = 3$

d. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{5x} = 1$

e. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{x} = \lim\limits_{x\to 0} \underbrace{\dfrac{\sin 5x}{x} \cdot \dfrac{5}{5}}_{\to 1} = 5$

f. $\lim\limits_{x\to -5} \dfrac{x^2+2x-15}{x^2-2x-35} = \lim\limits_{x\to -5} \dfrac{(x+5)(x-3)}{(x+5)(x-7)}$

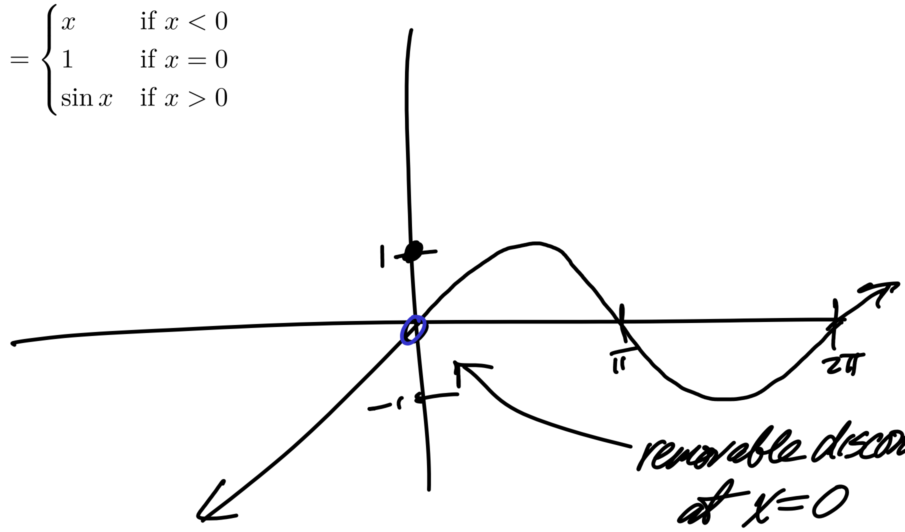
$= \lim\limits_{x\to -5} \dfrac{x-3}{x-7}$

$= \dfrac{-8}{-12}$

$= 2/3$

2. Graph each of the following functions. Find all discontinuities. For each one, write down the type of discontinuity and the left and right limits. If the discontinuity is removable, what should the function value be?

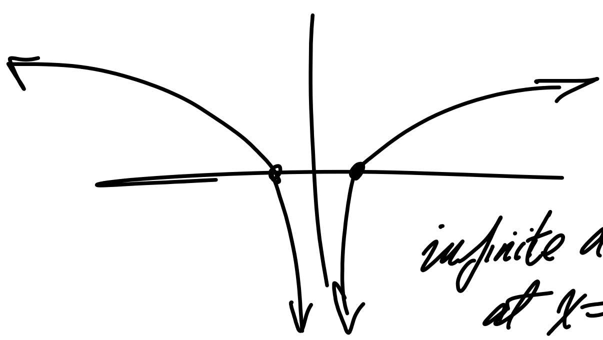
$$g(x) = \begin{cases} x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ \sin x & \text{if } x > 0 \end{cases}$$



removable discontinuity at $x=0$

$g(0)$ should be 0 (since $\lim_{x \to 0} g(x) = 0$)

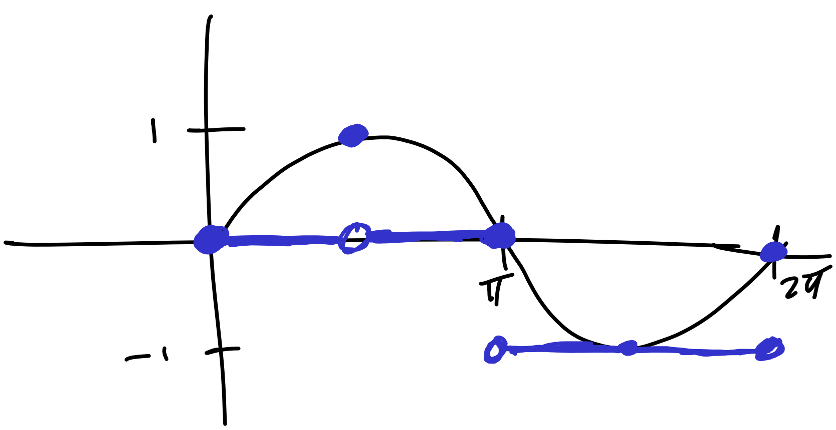
$h(x) = \ln(|x|)$

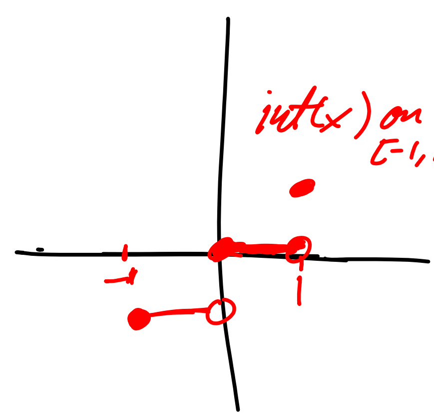


infinite discontinuity at $x=0$

$\lim_{x \to 0^{\pm}} h(x) = -\infty$

Challenge: $i(x) = \text{int}(\sin x)$





3. For the following functions find the derivative using a limit definition.

a. $f(x) = x^2 + 2x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) + 1 - (x^2 + 2x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h}$$

$$= \lim_{h \to 0} 2x + 2 + h$$

$$= 2x + 2$$

b. $g(x) = x^{-2}$

let's use: $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$

$$= \lim_{x \to a} \left(\frac{1}{x^2} - \frac{1}{a^2}\right) \frac{1}{x-a}$$

$$= \lim_{x \to a} \frac{a^2 - x^2}{x^2 a^2 (x-a)}$$

$$= \lim_{x \to a} \frac{-(x+a)(x-a)}{x^2 a^2 (x-a)}$$

$$= \lim_{x \to a} \frac{-(x+a)}{x^2 a^2}$$

$$= \frac{-2a}{a^4} = -2a^{-3}$$

$g'(a) = -2a^{-3}$ for any a

$\Rightarrow g'(x) = -2x^{-3}$

4. Using the various rules for differentiation, calculate the derivatives of the following functions.

a. $f(x) = x^2 + 2x + 1$

$$f'(x) = 2x+2$$

b. $g(x) = (x + 1)^2$. Use the product rule: $(x + 1)^2 = (x + 1)(x + 1)$. Verify your answer with Problems 3a and 4a.

$$g'(x) = 1(x+1) + (x+1)1$$
$$= 2\cdot(x+1)$$
$$= 2x+2$$

c. $h(x) = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x.$

$$h'(x) = 3x^3 + 2x^2 + x + 1$$

d. $k(x) = \frac{3x^2+2x+1}{4x+5}$. Here's the formula for the quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$k'(x) = \frac{(6x+2)(4x+5) - (3x^2+2x+1)(4)}{(4x+5)^2}$$
$$= \frac{24x^2+38x+10-12x^2-8x-4}{(4x+5)^2}$$
$$= \frac{12x^2+30x+6}{(4x+5)^2}$$

e. $m(x) = x^{-2} = \frac{1}{x^2}$. Do this first by using the quotient rule on $\frac{1}{x^2}$. Then verify your answer with the power rule and 3b.

quotient rule:

$$m'(x) = \frac{0\cdot(x^2) - 1(2x)}{(x^2)^2}$$
$$= -2x^{-3}$$

power rule

$$m(x) = x^{-2}$$
$$\Rightarrow m'(x) = -2x^{-3}$$

5. Let $f(t) = t^3 - 4t$.

a. Find all zeros, maxima and minima of f.

zeros at $t = 0, \pm 2$

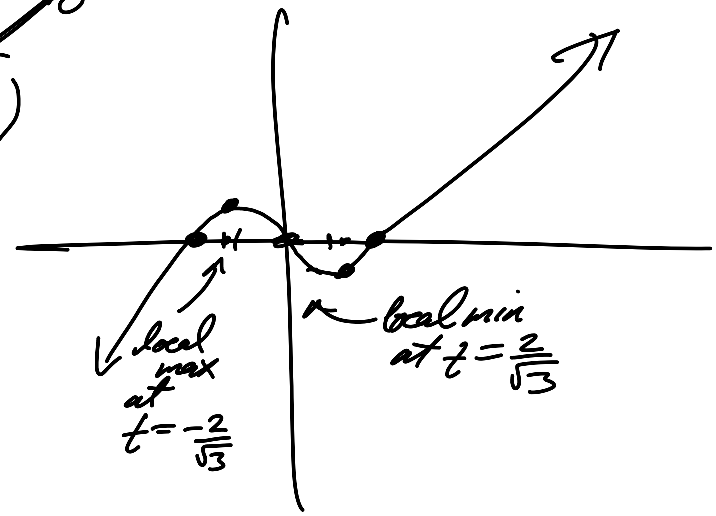
$f(t) = t(t^2-4) = t(t+2)(t-2)$

$f'(t) = 3t^2 - 4$

$f'(t) = 0 \Rightarrow t = \pm\sqrt{\frac{4}{3}}$

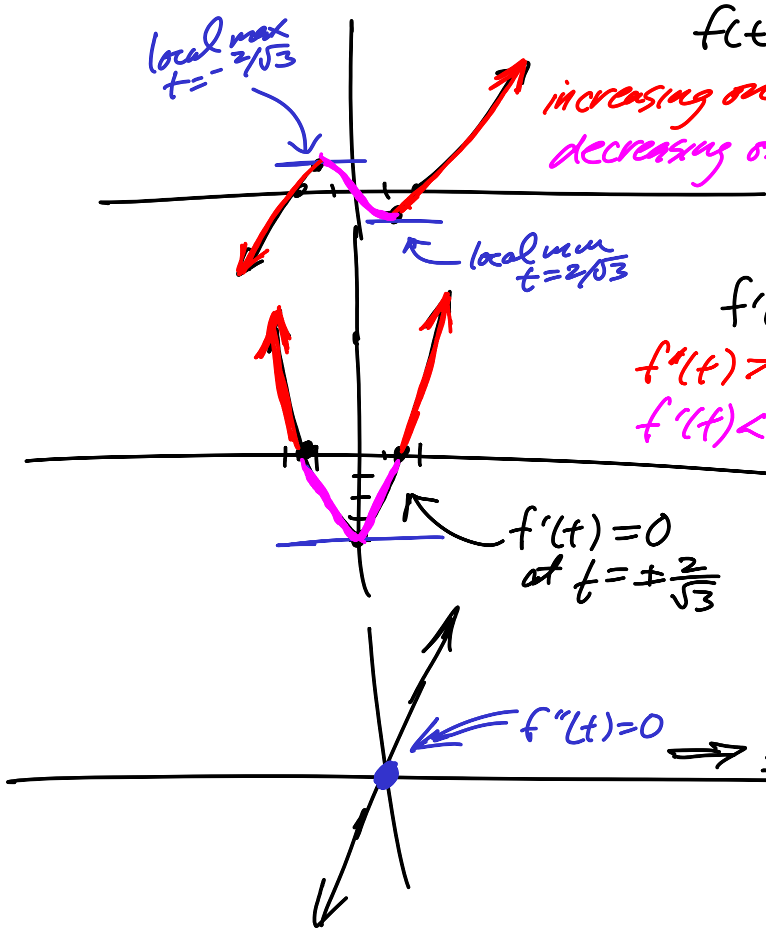
$= \pm \frac{2}{\sqrt{3}}$

no global min/max $\quad f''(t) = 6t$



b. Graph $f(t), f'(t), f''(t)$ (align the graphs in a column).

- On the graph of $y = f(t)$, indicate zeros, maxima/minima and intervals of increasing/decreasing. If $f(t)$ represents the height of an object at time t, how do you interpret these points/intervals?
- On the graph of $y = f'(t)$, indicate the zeros, and intervals where f' is positive/negative. How do you interpret these points/intervals?
- Where is the second derivative zero, and how do you interpret this on the graph of $y = f'(t)$?



$f(t) = t(t-2)(t+2)$

increasing on $(-\infty, -\frac{2}{\sqrt{3}})$ and $(\frac{2}{\sqrt{3}}, +\infty)$: going up

decreasing on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$: going down

$f'(t) = 3t^2 - 4$

$f'(t) > 0$ on $(-\infty, -2/\sqrt{3})$ and $(2/\sqrt{3}, +\infty)$: going up

$f'(t) < 0$ on $(-\frac{2}{\sqrt{3}}, 2/\sqrt{3})$: going down

$f''(t) = 6t$

$\Rightarrow$ slope of $f'(t)$ is 0 at $t = 0$; $f'(t)$ has a local min at $t = 0$