KEY

Unit 9 Group Work 2
PCHA 2021-22 / Dr. Kessner

No calculator! Have fun!

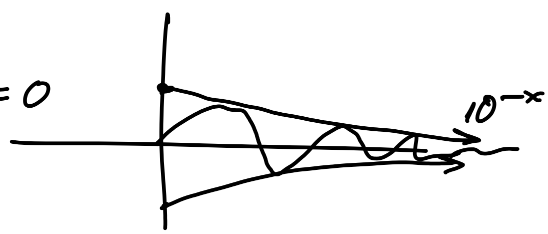
1. Evaluate the following limits, evaluating left and right side limits where applicable.

a. $\lim\limits_{x\to 0} x \csc \dfrac{x}{3}$ $= \lim\limits_{x\to 0} \dfrac{\dfrac{x \cdot \frac{1}{3}}{\sin(x/3)}}{\frac{1}{3}} = 3$

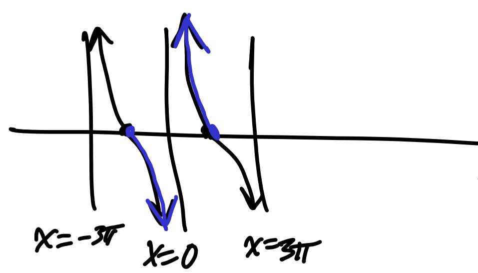
$\to 1$

b. $\lim\limits_{x\to 0} x \sin \dfrac{x}{3}$ $= 0$

c. $\lim\limits_{x\to \infty} 10^{-x} \sin \dfrac{x}{3}$ $= 0$



d. $\lim\limits_{x\to 0} \cot \dfrac{x}{3}$



$\lim\limits_{x\to 0^-} \cot \dfrac{x}{3} = -\infty$

$\lim\limits_{x\to 0^+} \cot \dfrac{x}{3} = +\infty$

e. $\lim\limits_{x\to 0} \dfrac{\sin(\pi + x) - \sin(\pi)}{x}$. $= \dfrac{d}{dx}(\sin x)(\pi)$

$= \cos \pi$

$= -1$

2. a. Find the derivative of $f(x) = \cos 2x$ using a limit definition. Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$.

Hint: Use the sum angle formula $\cos(u+v) = \cos u \cos v - \sin u \sin v$, but don't use the double angle formula.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(2(x+h)) - \cos 2x}{h}$$

$$= \lim_{h\to 0} \left(\cos 2x \cos 2h - \sin 2x \sin 2h - \cos 2x\right)\frac{1}{h}$$

$$= \lim_{h\to 0} \left[\cos 2x \left(\frac{\cos 2h - 1}{h}\right) - \sin 2x \frac{\sin 2h}{h}\right]$$

$\to 0$ $\to 2$

$$= -2\sin 2x$$

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

b. Find the derivative of $g(x) = \frac{1}{x}$ using the limit definition:

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a}$$

$$g'(a) = \lim_{x\to a} \frac{\left(\frac{1}{x} - \frac{1}{a}\right)}{x-a}$$

$$= \lim_{x\to a} \frac{1}{x-a}\left[\frac{a-x}{xa}\right]$$

$$= \lim_{x\to a} \frac{-1}{xa}$$

$$= -\frac{1}{a^2}$$

$$\boxed{g'(a) = -\frac{1}{a^2}} \longrightarrow \boxed{g'(x) = -\frac{1}{x^2}}$$

3. Using the various rules for differentiation, calculate the derivatives of the following functions.

a. $p(x) = e^{\sin x}$

$p'(x) = e^{\sin x}(\cos x)$

b. $q(x) = \sin^2 x + \cos^2 x$ (Practice using power and chain rules!)

$q'(x) = 2\sin x(\cos x) + 2\cos x(-\sin x)$
$= 0$

c. $r(x) = \sin^4 x - \cos^4 x$

$r'(x) = 4\sin^3 x \cos x - 4\cos^3 x(-\sin x)$
$= 4\sin^3 x\cos x + 4\cos^3 x \sin x$
$\left(= 4\sin x\cos x(\cos^2 x + \sin^2 x)\right.$
$\left. = 4\sin x\cos x \right)$

d. $s(x) = -\cos 2x$ (Notice that $s'(x) = r'(x)$. Challenge: verify that $r(x) = s(x)$.)

$s'(x) = +2\sin 2x$
$(= 4\sin 2x\cos 2x)$

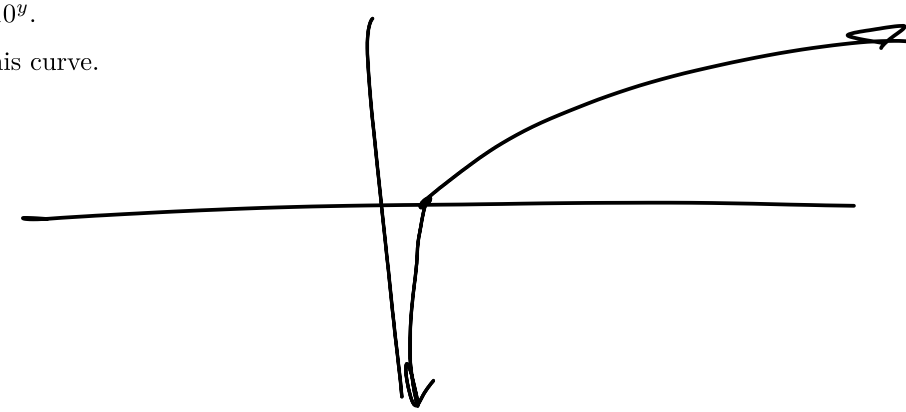
$\sin^4 x - \cos^4 x$
$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$
$= -(\cos^2 x - \sin^2 x)$
$= -\cos 2x$ ✓

e. $t(x) = 2^{\sin x^2}$

$t'(x) = 2^{\sin x^2} \ln 2(\cos x^2)(2x)$

4. Consider the curve $x = 10^y$.

a. Sketch the graph of this curve.



$x = 10^y$

$y = \log_{10} x$

b. Find $\frac{dy}{dx}$ (in terms of x and y) by implicit differentiation.

$x = 10^y$

$1 = 10^y \ln 10 \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{10^y \ln 10}$

c. Solve for y in terms of x. $y = \log_{10} x$

d. Find $\frac{dy}{dx}$ using the expression for y you found above.

$\frac{dy}{dx} = \frac{1}{x \ln 10}$

e. Verify that these two formulas for $\frac{dy}{dx}$ are the same.

$x = 10^y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{10^y \ln 10} = \frac{1}{x \ln 10}$ ✓

5. Suppose you have 128 kg of ^{14}C, which has a half-life of 5730 years.

a. Write an equation to model the amount $A(t)$ of ^{14}C as a function of time.

t	$A(t)$
0	128
5730	64
$5730 \cdot 2$	32

$$A(t) = 128 \left(\frac{1}{2}\right)^{\underbrace{t/5730}_{\text{\# of half lives}}}$$

b. Find the average rate of change in the amount over the first 5 half-lives ($5 \cdot 5730$ years). Use a calculator to get approximate values.

$$\frac{A(5) - A(0)}{5 \cdot 5730} = \frac{4 - 128}{5 \cdot 5730} = \frac{-124}{5 \cdot 5730} \approx -.00432$$

c. Find $A'(t)$.

$$A'(t) = 128 \left(\frac{1}{2}\right)^{t/5730} \cdot \ln\left(\frac{1}{2}\right)\left(\frac{1}{5730}\right)$$

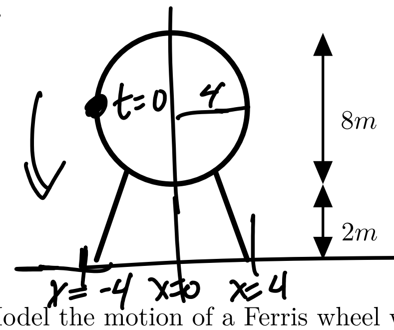
d. Calculate the rate of change (exact) at $t = 0$, $t = 2 \cdot 5730$, and $t = 5 \cdot 5730$ years. Use a calculator to get approximate values.

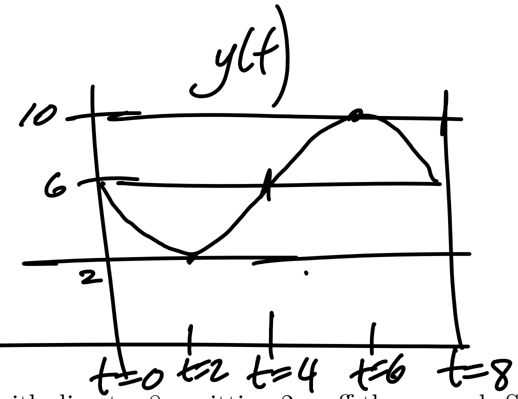
$$A'(0) = \frac{128 \ln\left(\frac{1}{2}\right)}{5730} \approx -.0155$$

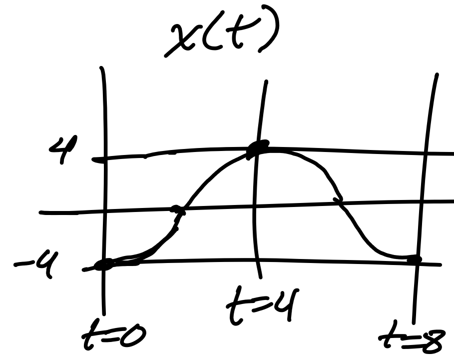
$$A'(2) = \frac{128 \ln\left(\frac{1}{2}\right)}{5730} \cdot \frac{1}{4} \approx -.00387$$

$$A'(5) = \frac{128 \ln\left(\frac{1}{2}\right)}{5730} \cdot \frac{1}{32} \approx -.000484$$

6.







Model the motion of a Ferris wheel with diamter 8m, sitting 2m off the ground. Suppose you start ($t = 0$) at the 9 o'clock position (furthest left on diagram), traveling counter-clockwise, and that the period is 8 minutes.

a. Write parametric equations $x(t)$ and $y(t)$ to model the position as a function of time.

$$x(t) = -4\cos\left(\frac{2\pi t}{8}\right) = -4\cos\left(\frac{\pi}{4}t\right)$$

$$y(t) = 6 - 4\sin\left(\frac{2\pi t}{8}\right) = 6 - 4\sin\left(\frac{\pi}{4}t\right)$$

b. Find $x'(t)$ and $y'(t)$.

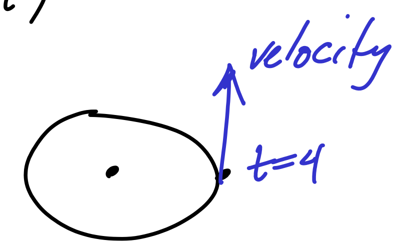
$$x'(t) = 4\sin\left(\frac{\pi}{4}t\right)\cdot\frac{\pi}{4} = \pi\sin\left(\frac{\pi}{4}t\right)$$

$$y'(t) = -4\cos\left(\frac{\pi}{4}t\right)\cdot\frac{\pi}{4} = -\pi\cos\left(\frac{\pi}{4}t\right)$$

c. Evaluate $x'(t)$ and $y'(t)$ at the rightmost position. $\Rightarrow t=4$

$$x'(4) = \pi\sin\pi = 0$$

$$y'(4) = -\pi\cos\pi = \pi$$



d. Find $x''(t)$ and $y''(t)$.

$$x''(t) = \frac{\pi^2}{4}\cos\left(\frac{\pi}{4}t\right)$$

$$y''(t) = +\frac{\pi^2}{4}\sin\left(\frac{\pi}{4}t\right)$$

e. Evaluate $x''(t)$ and $y''(t)$ at the rightmost position.

$$x''(4) = \frac{\pi^2}{4}\cos\pi = -\frac{\pi^2}{4}$$

$$y''(4) = +\frac{\pi^2}{4}\sin\pi = 0$$